Sydney Girls High School

November 2015

MATHEMATICS
EXTENSION 1

Year 12
ASSESSMENT TASK 1 FOR HSC 2016

Time Allowed: 60 minutes + 5 minutes reading time

Topics: Integration, Trigonometric Functions II, Parametric Equations

General Instructions:

- **There are Seven (7) questions each worth 8 marks**
- **Attempt all questions.**
- **Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.**
- **Start each question on a new page.**
- **Write on one side of the paper only.**
- **Diagrams are NOT to scale.**
- **Board-approved calculators may be used.**
- **Mathematics reference sheet is provided.**

TOTAL: 56 MARKS

Student's Name: ……………………………….**Teacher Name:** ……………………..

QUESTION 1 **Start a new page** **Marks**

(a) Find $\int \frac{dx}{(2x-1)^4}$ 2

(b) Evaluate $\int_{-2}^{3} (5-x^2) \ dx$ 2

(c) Find the exact value of $\cos 75^\circ$. 2

(d) Find the Cartesian equation of a function whose parametric equations are $x = 2t$ and $y = t^2 - 1$. 2

Question 2 **Start a new page** **Marks**

(a) Given that $\int_1^2 f(x)\ dx = 3$ and $f(x) + f(-x) = 0$ find $\int_{-2}^{-1} f(x)\ dx$. 1

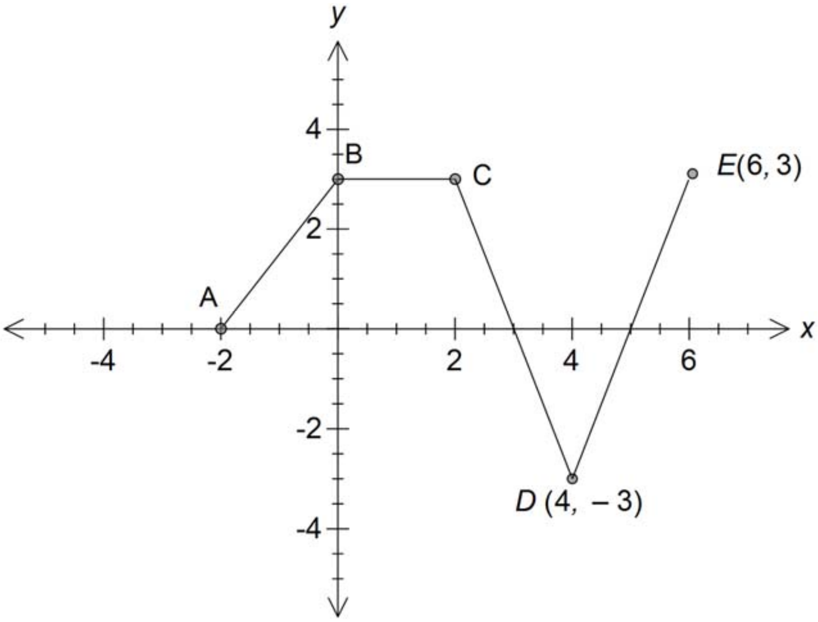
(b) Find the area bounded by the curve $y = 9 - x^2$ and the x axis. 2

(c) Given that the exact value of $\sin x$ is $\frac{7}{15}$ and the exact value of $\cos y$ is $\frac{1}{\sqrt{3}}$ find the exact value of sin $(x + y)$ where x and y are acute. 3

(d) The point $(8t, 4t^2)$ lies on the parabola $x^2 = 16y$. Find the equation of the chord whose endpoints have parameters 2 and -1. 2

Question 3 **Start a new page** **Marks**

(a)



Evaluate

(i) $\int_0^2 f(x)\ dx$ 1

(ii) $\int_3^5 f(x)\ dx$ 1

(iii) $\int_{-2}^6 f(x)\ dx$ 1

(b) Solve $\sin 2x - \sin x = 0$ for $0 \le x \le 2\pi$. 3

(c) For the parabola $x = 6t, y = 3t^2$ find the equation of the normal at $t = 2$. 2

Question 4 **Start a new page** **Marks**

(a) Evaluate $\int_0^2 3^x \, dx$ using Simpson's rule with 5 function values. 2

Answer correct to 2 decimal places.

(b) (i) Express $3\cos 2x - 4\sin 2x$ in the form $\mathrm{R}\cos(2x+\alpha)$. 2

(ii) Hence solve $6\cos 2x - 8\sin 2x = 2$ for $0^\circ \le x \le 360^\circ$. 2

(c) From an external point two tangents are drawn to the parabola $y = x^2$.

Find the co-ordinates of the external point if the equation of the chord of contact of the two tangents is $y = 4x + 3$. 2

Question 5 **Start a new page** **Marks**

(a) The region bounded by $y = 3$, $x = y^2$ and the y axis is rotated around the x axis.

(i) Sketch the region above on a number plane. 1

(ii) Find the exact volume of the solid of rotation. 2

(b) Prove that $\tan\frac{x}{2} = \dfrac{\sin x - \cos x + 1}{\sin x + \cos x + 1}$ 2

(c) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$

(i) Show that $m_{PQ} = \dfrac{p+q}{2}$. 1

(ii) If $pq = -1$ show that PQ is a focal chord. 2

Question 6 Start a new page **Marks**

(a) (i) Find the points of intersection of $y = x^2 - 3x + 1$ and $y = 2x - 5$. 2

(ii) Find the area bounded by $y = x^2 - 3x + 1$ and $y = 2x - 5$. 2

(b) Solve $3\sin\theta + 2\cos\theta = 1$ by using the t method for $0^\circ \leq \theta \leq 360^\circ$. 2

(c) Find the equation of the locus of the midpoint of PQ where P is the point $(2ap, ap^2)$ and Q is the point $(6ap, 9ap^2)$ on the parabola $x^2 = 4ay$. 2

Question 7 Start a new page **Marks**

(a) Find the area bounded by the curve $y = 9x - x^3$ and the x axis. 2

(b) Find the exact value of $\tan\dfrac{3\pi}{8}$. 3

(c) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$. 3

The tangents at P and Q meet at R, and R lies on the parabola $x^2 = -4ay$.

Show that $p^2 + q^2 + 6pq = 0$.

Question 1 - Maths Ext 1 – Assessment Task 1 Year 12

(a) $\int \frac{dx}{(2x-1)^4}$

$\int (2x-1)^{-4} dx$

$\frac{(2x-1)^{-3}}{-3\times 2} + c$

$-\frac{(2x-1)^{-3}}{6} + c$

$-\frac{1}{6(2x-1)^3} + c$

(a) need to be careful adding a one to a negative number (2)

(b) $\int_{-2}^{3} (5-x^2)\,dx$

$\left[5x-\frac{x^3}{3}\right]_{-2}^{3}$

$\left[5(3)-\frac{(3)^3}{3}\right]-\left[5(-2)-\frac{(-2)^3}{3}\right]$

$[15-9]-[-10+\frac{8}{3}]$

$= \frac{40}{3}$

(2)

(c) cos 75

cos (30 + 45)

cos 30 × cos 45 − sin 30 sin 45

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

or

$\frac{\sqrt{6}-\sqrt{2}}{4}$ (2)

(d) $x = 2t$ $y = t^2 - 1$

$t = \frac{x}{2}$ $t^2 = y + 1$

$t^2 = \frac{x^2}{4}$

$\frac{x^2}{4} = y + 1$

$x^2 = 4(y+1)$

(2)

(c) Few students did not use correct addition of angles formula.

$\boxed{Q_2}$

a) $\int_1^2 f(x)\,dx = 3$; $f(x) + f(-x) = 0$

$f(x) = -f(-x)$

$$\int_{-2}^{-1} f(x)\,dx = -\int_{-1}^{-2} f(x)\,dx$$

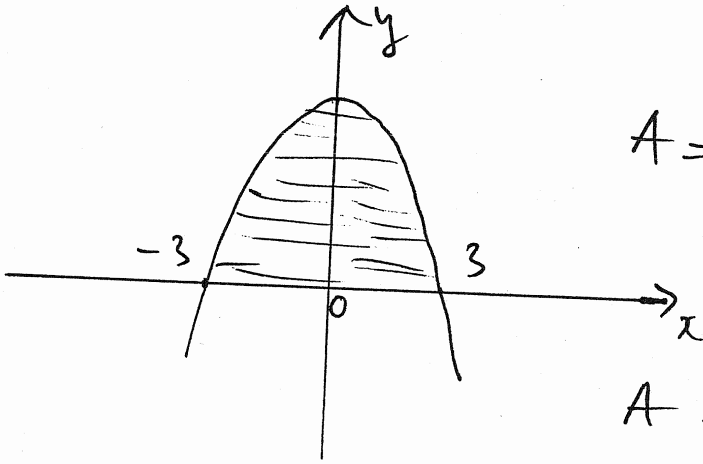
$$= -\int_1^2 f(-x)\,dx = -3 \checkmark$$

Some students could not regeconise this is an odd function.

b) $y = 9 - x^2$

x - intercepts : $(3-x)(3+x) = 0$

$x = \pm 3$



$$A = \int_{-3}^{3} (9-x^2)\,dx$$

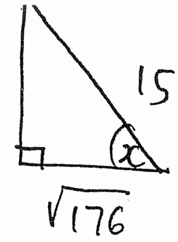
$$A = 2\int_0^3 (9-x^2)\,dx$$

$$A = 2\left[9x - \frac{x^3}{3}\right]_0^3$$

$$A = 2\left[9\times3 - \frac{3^3}{3} - 0\right]$$

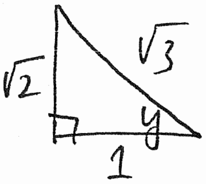
$A = 36$ units^2

c) $\sin x = \frac{7}{15}$



$\cos x = \frac{\sqrt{176}}{15}$ ✓

$\cos y = \frac{1}{\sqrt{3}}$



$\sin y = \frac{\sqrt{2}}{\sqrt{3}}$

$$\sin(x+y) = \sin x \cdot \cos y + \cos x \sin y$$

$$= \frac{7}{\sqrt{15}} \cdot \frac{1}{\sqrt{3}} + \frac{\sqrt{176}}{15} \cdot \frac{\sqrt{2}}{\sqrt{3}}$$ ✓ ✓

$$= \frac{7 + \sqrt{352}}{15\sqrt{3}} = \frac{7\sqrt{3} + 4\sqrt{66}}{45}$$

* Some students could not find the ratios $\cos x$ and $\sin y$, hence could not evaluate $\sin(x+y)$.

d) At $A(8t, 4t^2)$

At $t = 2 \therefore P(16, 16)$

$t = -1 \therefore Q(-8, 4)$

Eq of the Chord PQ

$$y - y_1 = m(x - x_1)$$

$$y - 16 = \frac{4-16}{-8-16}(x-16)$$

$$x - 2y + 16 = 0$$

Alternate Method

$y = \frac{p+q}{2}x - a$

$y = \frac{2-1}{2}x - 4(2)(-$

$y = \frac{1}{2}x + 8$

$x - 2y + 16 = 0$

Ext 1 2015 Assessmat 1

3a) i)

$$\int_0^2 f(x)\,dx = 2\times 3$$
$$= 6$$ ✓

Many students calculated the the area instead of Evaluating the Integral.

ii) $$\int_3^5 f(x) = -\frac{1}{2}\times 2\times 3$$
$$= -3$$ ✓

iii) $$\int_{-2}^{6} f(x)\,dx = \frac{1}{2}\times 2\times 3 + 6 + \frac{1}{2}\times 3\times 1 - 3 + \frac{1}{2}\times 3\times 1$$
$$= 9$$ ✓

b) $2\sin x\cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$ ✓

Some students didn't have all the solutions

$\sin x = 0$

$x = 0, \pi, 2\pi$ ✓

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

$\therefore x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ ✓

c) $t = 2$

This question was done well

$\therefore x = 12, \quad y = 12$

✓ ✓

$m_1 = 2 \quad m_2 = -\frac{1}{2}$

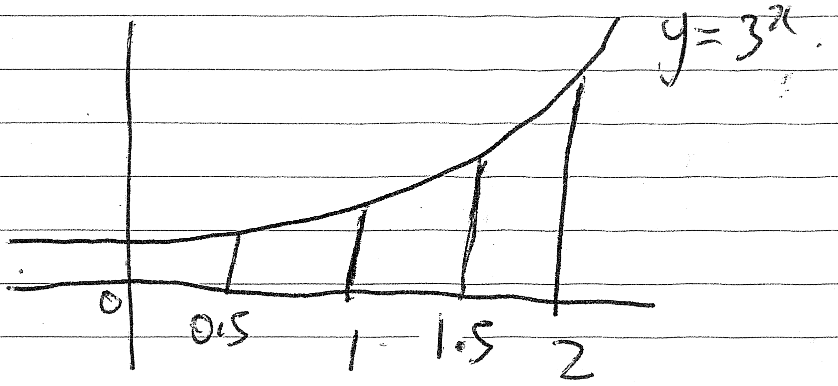
$y - 12 = -\frac{1}{2}(x - 12)$

$2y - 24 = -x + 12$ $\quad \therefore \boxed{x + 2y - 36 = 0}$

ASSEMENT TASK 1 2015 Ext 1

Q4.

(a)



$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

$$\int_0^1 3^x\,dx + \int_1^2 3^x\,dx$$

$$= \frac{1}{6}\left[3^0+4(3^{0.5})+3^1\right]+\frac{1}{6}\left[3^1+4(3^{1.5})+3^{2}\right]$$

$$= \frac{1}{6}(4+4\sqrt{3})+\frac{1}{6}(12+12\sqrt{3})$$

$$= \frac{1}{6}(16+16\sqrt{3})$$

$$= \frac{8}{3}+\frac{8\sqrt{3}}{3} \approx 7.29$$

If you can't remember Simpson's rule, it's on the reference sheet.

(b) (i) $R\cos(2x+\alpha) = R\cos\alpha\cos 2x - R\sin\alpha\sin 2x$

$3\cos 2x - 4\sin 2x$

$R\sin\alpha = 4$

$R\cos\alpha = 3$

$\tan\alpha = \frac{4}{3}$

$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$

$\approx 53^\circ 8'$

$R^2\sin^2\alpha + R^2\cos^2\alpha = 16+9$

$R = 5$

$$5\cos(2x + 53^\circ 8')$$

(ii) $6\cos 2x - 8\sin 2x = 2$

$3\cos 2x - 4\sin 2x = 1.$

$5\cos(2x+\alpha) = 1$

Let $\beta = \cos^{-1}\left(\frac{1}{5}\right)$.

$2x+\alpha = \beta,\ 360^\circ-\beta,\ 360^\circ+\beta,\ 720^\circ-\beta.$

$2x = \beta-\alpha,\ 360^\circ-\beta-\alpha,\ 360^\circ+\beta-\alpha,\ 720^\circ-\beta-\alpha$

$x = \frac{\beta-\alpha}{2},\ \frac{360^\circ-\beta-\alpha}{2},\ \frac{360^\circ+\beta-\alpha}{2},\ \frac{720^\circ-\beta-}{2}.$

$= 12^\circ 40',\ 114^\circ 12',\ 192^\circ 40',\ 294^\circ 12'$

Some students only found solutions $0^\circ \leq x \leq 180^\circ$.

(c) $y = x^2$

$4(\frac{1}{4})y = x^2$

$a = \frac{1}{4}$

Chord of Contact

$x_0 x = 2a(y + y_0)$

$x_0 x = \frac{1}{2}(y + y_0)$.

Given $y = 4x + 3$

$4x = y - 3$

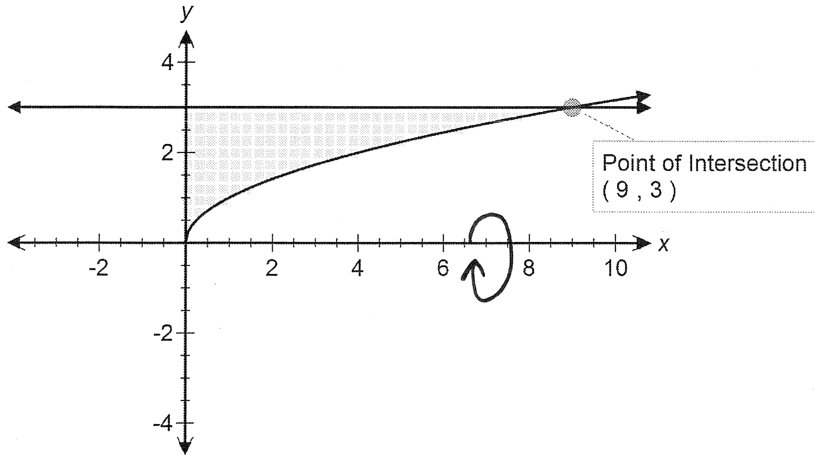
$2x = \frac{1}{2}(y - 3)$.

Ext. Point $(2, -3)$.

Some student tried to solve the two equations simultaneously. While it is possible to do this question that way it proved to be too difficult.

Question 5 (8 Marks)

a) i)



①

ii) $V = \pi r^2 h - \pi \int y^2 dy$

$= \pi \times 3^2 \times 9 - \pi \int_0^9 x \, dx$

$= 81\pi - \pi \left[\frac{x^2}{2}\right]_0^9$

$= 81\pi - \frac{81\pi}{2}$

$= 40\frac{1}{2}\pi$ units^3.

②

Many students could not work out the point of intersection which then determined the limits of integration.

OR

$V = \pi \int [f(x)^2 - g(x)^2] \, dx$.

$= \pi \int_0^9 [(3)^2 - y^2] \, dx$

$= \pi \int_0^9 (9 - x) \, dx$

$= \pi \left[9x - \frac{x^2}{2}\right]_0^9$

$= \pi \left[81 - \frac{81}{2}\right]$

$= \frac{81}{2}\pi$ units^3.

<u>MOST</u> students lost a mark because of the <u>incorrect method</u> of finding the volume of the shaded region.

b) $$\text{RHS} = \frac{\sin x - \cos x + 1}{\sin x + \cos x + 1}$$

$$= \left(\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} + 1\right) \div \left(\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1\right)$$

$$= \left(\frac{2t - 1 + t^2 + 1 + t^2}{1+t^2}\right) \div \left(\frac{2t + 1 - t^2 + 1 + t^2}{1+t^2}\right)$$

②

$$= \left(\frac{2t^2 + 2t}{1+t^2}\right) \times \left(\frac{1+t^2}{2t+2}\right)$$

$$= \frac{2t(t+1)}{2(t+1)}$$

$$= t$$

$$= \text{LHS}.$$

Many students could not complete this proof and lost one mark.

OR. (A few students proved as follows):

$$\text{RHS} = \frac{\sin x - \cos x + 1}{\sin x + \cos x - 1}$$

$$= \frac{\sin x + (1 - \cos x)}{\sin x + (-1 + \cos x)}$$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2} + 2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2} + 2\cos^2\frac{x}{2}}$$

$$= \frac{2\sin\frac{x}{2}\ \cancel{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}}{2\cos\frac{x}{2}\ \cancel{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)}}$$

$$= \tan\frac{x}{2}$$

$$= t = \text{LHS}.$$

c) i) $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{\cancel{a}(\cancel{p-q})(p+q)}{2\cancel{a}(\cancel{p-q})}$ ①

$= \dfrac{p+q}{2}$

ii) Equation of PQ:

$$y - ap^2 = \frac{p+q}{2}(x - 2ap)$$

$$2y - 2ap^2 = px - 2ap^2 + qx - 2apq$$

$$y = \left(\frac{p+q}{2}\right)x - apq$$

step① If $pq = -1$

then: $y = \left(\dfrac{p+q}{2}\right)x + a$

step② Substitute focus $(0, a)$: ②

LHS $= y$ RHS $= \left(\dfrac{p+q}{2}\right) \times 0 + a$

$= a$ $= a$

LHS = RHS

∴ focus satisfies eqn of chord PQ if $pq = -1$.

Students lost one mark if they showed step② before step①.

6 (a) (i) $x^2 - 3x + 1 = 2x - 5$

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$x = 3, 2$

when $x = 3$, $y = 2 \times 3 - 5$
$= 1$

'' $x = 2$, $y = 2 \times 2 - 5$
$= -1$

When finding points both co-ordinates need to be found.

(ii) $\int_2^3 2x - 5 - (x^2 - 3x + 1)\,dx$

$= \int_2^3 (-x^2 + 5x - 6)\,dx$

$= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 6x\right]_2^3$

$= -9 + \frac{45}{2} - 18 + \frac{8}{3} + 10 + 12$

$= \frac{1}{6}$

(b) $3 \times \frac{2t}{1+t^2} + 2 \times \frac{1-t^2}{1+t^2} = 1$

$6t + 2 - 2t^2 = 1 + t^2$

$0 = 3t^2 - 6t - 1$

$t = \frac{6 \pm \sqrt{6^2 - 4 \times 3 \times -1}}{2 \times 3}$

$= \frac{6 \pm \sqrt{48}}{6}$

$= \frac{6 \pm 4\sqrt{3}}{6}$

$= \frac{3 \pm 2\sqrt{3}}{3}$

$\tan \frac{\theta}{2} = 65°, 171°$

$\therefore \theta = 130°, 342°$

$\frac{\theta}{2}$ has to be between 0° and 180°.

(c) $x = \frac{2ap + 6ap}{2}$

$= 4ap$

$p = \frac{x}{4a}$

$y = ap^2 + 9ap^2$

$= \frac{10ap^2}{2}$

$= 5ap^2$

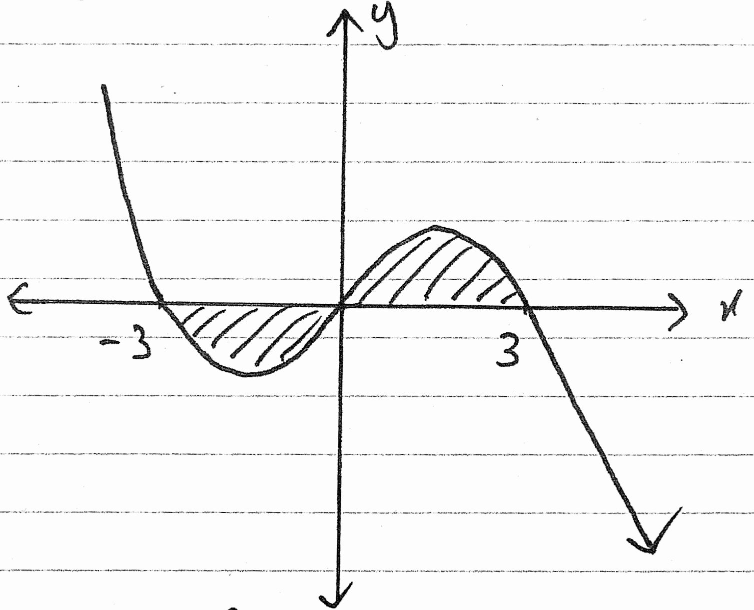
$= 5a \times \left(\frac{x}{4a}\right)^2$

$= \frac{5x^2}{16a}$

There must be no "p" in the final answer.

QUESTION 7

(a) $y = x(9-x^2)$



$$A = 2\int_0^3 (9x - x^3)\,dx$$

This question was done well by most students. Some students did not identify the correct intercepts on the grap or didn't identify the correct region of interest.

$$= 2\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3$$

$$= 2\left(\frac{81}{2} - \frac{81}{4} - (0)\right)$$

$$\therefore A = 40.5 \text{ units}^2$$

(b) Consider $t = \tan\frac{\theta}{2}$ where $\theta = \frac{3\pi}{4}$

$$\therefore t = \tan\frac{3\pi}{8}$$

$$\tan\theta = \frac{2t}{1-t^2} \qquad \therefore \tan\frac{3\pi}{4} = \frac{2t}{1-t^2}$$

$$-1 = \frac{2t}{1-t^2}$$

Many students failed to make the connection to the t-formula on this question.

$$-1 + t^2 = 2t$$

$$t^2 - 2t - 1 = 0$$

$$(t-1)^2 = 2$$

(b) Continued

$\therefore t = 1 \pm \sqrt{2}$

To get this question out fully, you need to identify that there is only one answer and eliminate the minus sign.

$\tan \frac{3\pi}{8} > 0$ $\left(\frac{3\pi}{8}\right.$ in quadrat 1)

$\therefore \tan \frac{3\pi}{8} = 1 + \sqrt{2}$

(c) Tangent at $P(2ap, ap^2)$

$m_T = p$

$\therefore y - ap^2 = p(x - 2ap)$

$y = px - ap^2$

Tangent at Q: $y = qx - aq^2$

R: $px - ap^2 = qx - aq^2$

$x(p-q) = a(p^2 - q^2)$

$= a(p-q)(p+q)$

$\therefore x = a(p+q)$

$y = p \times a(p+q) - ap^2$

$= ap^2 + apq - ap^2 = apq$

$\therefore$ R is $(a(p+q), apq)$

Finding the coordinates of R should have been fairly straight forward as a parametrics question though it seems that some students may have struggled with time at the end of the paper.

Since R lies on $x^2 = -4ay$

$(a(p+q))^2 = -4a(apq)$

$a^2(p^2 + 2pq + q^2) = -4a^2(pq)$

$\therefore p^2 + 2pq + q^2 = -4pq$

i.e. $p^2 + 6pq + q^2 = 0$

Proving the required result was difficult for students.